

An Explicit Formula of the Intrinsic Metric on the Sierpinski Gasket via Code Representation

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Abstract

The computation of the distance between any two points of the Sierpinski Gasket with respect to the intrinsic metric has already been investigated by several authors. In the literature there is not an explicit formula using the code space of the Sierpinski Gasket. In this paper, we give an explicit formula for the intrinsic metric on the Sierpinski Gasket via code representations of its points.

Keywords: Sierpinski Gasket, Code space, Intrinsic metric.

2000 AMS Classification: 28A80, 51F99

1. Introduction

The Sierpinski Gasket was described by W. Sierpinski in 1915, and then it became one of the typical examples of fractals. Sierpinski Gasket has been studied in fractal geometry for years (see for example [1, 7] for more information). It is well known that S is the attractor of the iterated function system $\{\mathbb{R}^2; f_0, f_1, f_2\}$ where

$$\begin{aligned} f_0(x, y) &= \left(\frac{1}{2}x, \frac{1}{2}y \right) \\ f_1(x, y) &= \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y \right) \\ f_2(x, y) &= \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4} \right). \end{aligned}$$

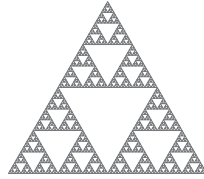


Figure 1. The Sierpinski Gasket as an attractor of an IFS.

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In [5, 6], the authors define S as follows: Let $P_0 = (0, 0)$, $P_1 = (0, 1)$ and $P_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Assume that $i_1 i_2 \dots i_n$ is the word of length n over the alphabet $X = \{0, 1, 2\}$ for any $i_1, i_2, \dots, i_n \in X$. For every such word, it is denoted the elementary sub-triangle of level n with vertices $f_{i_1}(P_0) \circ f_{i_2}(P_0) \circ \dots \circ f_{i_n}(P_0)$, $f_{i_1}(P_1) \circ f_{i_2}(P_1) \circ \dots \circ f_{i_n}(P_1)$ and $f_{i_1}(P_2) \circ f_{i_2}(P_2) \circ \dots \circ f_{i_n}(P_2)$ by $T_{i_1 i_2 \dots i_n}$. Then they define the Sierpinski Gasket as

$$S = \bigcup_{n \geq 0} T_n \text{ where } T_n = \bigcup_{s \in \{0,1,2\}^n} T_s.$$

As known, it can be constructed several metric structures on a set. But, a metric which is not take into consideration its internal structure is far from being applicable. For example, consider the restriction of Euclidean metric to S . According to this metric, the distance between a and b is l (see Figure 2). However, there is not any path between a and b on S with length l . For this reason, this metric is not meaningful on this special set. The intrinsic metric which is obtained by taking into account the paths on the structure, eliminates this discrepancy.

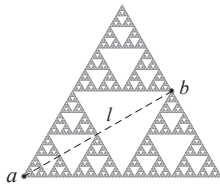


Figure 2. Distance between two points on S with respect to the Euclidean metric.

One can define the intrinsic metric on S such that

$$d(x, y) = \inf\{\delta \mid \delta \text{ is the length of a rectifiable curve in } S \text{ joining } x \text{ and } y\}$$

for $x, y \in S$ (for details see [2]).

In the several works, the intrinsic metric on the Sierpinski Gasket was constructed and defined in different ways since there exist different ways to construct (or define) the Sierpinski Gasket (for details see [3, 5, 6, 7]). For example in [5], it is given an alternative definition of the intrinsic metric on S as follows: Let $x, y \in S$ and let $\Delta_n(x), \Delta_n(y)$ be two elementary sub-triangles of level n where $x \in \Delta_n(x)$ and $y \in \Delta_n(y)$ for all $n \geq 0$. For every $n \geq 0$, the left lower vertices of $\Delta_n(x)$ and $\Delta_n(y)$ respectively. Then the authors define the intrinsic metric as

$$d(x, y) = \lim_{n \rightarrow \infty} \frac{d_n(x_n, y_n)}{2^n}$$

where $x, y \in S$.

R. Strichartz also defines the intrinsic metric in a different way by using barycentric coordinates (for details see [9]).

In [8], Romik tackle the discrete Sierpinski Gasket and define the metric giving the shortest distance on the points of this set using by the code spaces. Romik then compute the average distance between points on the Sierpinski Gasket using

the connection between the Tower of Hanoi problem and the discrete Sierpinski Gasket.

In this paper, we use code representations of the points of the Sierpinski Gasket to define the intrinsic metric. We note that the junction points of the Sierpinski Gasket have two different code representations. In this work, we give an explicit formula for the intrinsic metric on S such that the formula does not depend on the choice of the representations of the junction points as mentioned in Proposition 3.4.

2. Code representation on the Sierpinski Gasket

We first give a small brief about the coding process.

Let us denote the left-bottom part, the right-bottom part and the upper part of the Sierpinski Gasket by S_0, S_1 and S_2 respectively (see Figure 3).

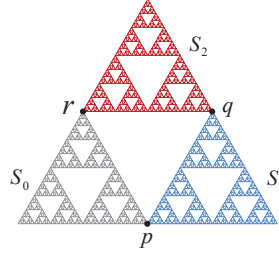


Figure 3. The sub-triangles S_0, S_1 and S_2 of S .

As shown in Figure 3, $S = S_0 \cup S_1 \cup S_2$, $S_0 \cap S_1 = \{p\}$, $S_1 \cap S_2 = \{q\}$ and $S_0 \cap S_2 = \{r\}$. Let $a_1 \in \{0, 1, 2\}$. Now similarly we denote the left-bottom part, the right-bottom part and the upper part of S_{a_1} by $S_{a_1 0}, S_{a_1 1}$ and $S_{a_1 2}$ respectively (see Figure 4).

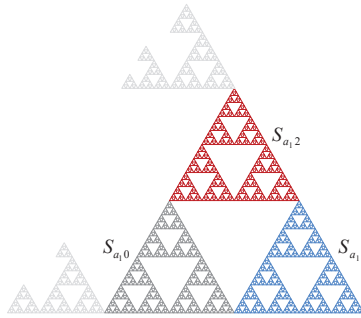


Figure 4. The sub-triangles $S_{a_1 0}, S_{a_1 1}$ and $S_{a_1 2}$ of S_{a_1} for $a_1 = 1$.

With the same argument, let $S_{a_1 a_2 \dots a_k}$ denote the smaller triangular pieces of S where $a_i \in \{0, 1, 2\}$ and $i = 1, 2, \dots, k$. For the sequence

$$S_{a_1}, S_{a_1 a_2}, S_{a_1 a_2 a_3}, \dots, S_{a_1 a_2 \dots a_n}, \dots,$$

it is obvious that $S_{a_1} \supset S_{a_1 a_2} \supset S_{a_1 a_2 a_3} \supset \dots \supset S_{a_1 a_2 \dots a_n} \supset \dots$ and the infinite intersection

$$\bigcap_{k=1}^{\infty} S_{a_1 a_2 \dots a_k}$$

is a singleton, say $\{a\}$ where $a \in S$. We denote the point $a \in S$ by $a_1 a_2 \dots a_n \dots$ where $a_n \in \{0, 1, 2\}$ and $n = 1, 2, \dots$. Note that, if $a \in S$ is the intersection point of any two sub-triangles of $S_{a_1 a_2 \dots a_k}$ (such a point is called a junction point of S) then a has two different representations such that $a_1 a_2 \dots a_k \beta \alpha \alpha \alpha \dots$ and $a_1 a_2 \dots a_k \alpha \beta \beta \beta \dots$ where $\alpha, \beta \in \{0, 1, 2\}$ (see Figure 5). Otherwise, a has a unique representation.

(For an alternative code space representation of the points of S , see [4].)

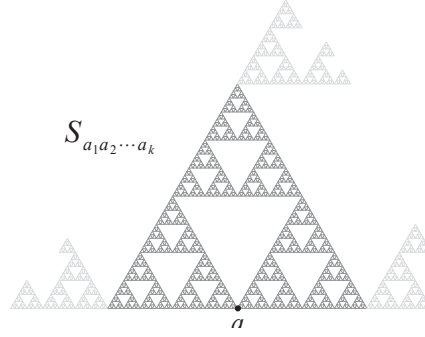


Figure 5. The point a for $\alpha = 0$ and $\beta = 1$.

3. Construction of the intrinsic metric on S

Let a and b be two different points of S whose representations are $a = a_1 a_2 \dots a_n \dots$ and $b = b_1 b_2 \dots b_n \dots$ respectively. Then there exists a natural number s such that $a_s \neq b_s$. Let

$$(3.1) \quad k = \min\{s \mid a_s \neq b_s, s = 1, 2, 3, \dots\}.$$

We then have $a \in S_{a_1 a_2 \dots a_{k-1} a_k}$ and $b \in S_{a_1 a_2 \dots a_{k-1} b_k}$. Without loss of generality, we assume that $a_k = 0$ and $b_k = 1$ which means $a \in S_{a_1 a_2 \dots a_{k-1} 0}$ and $b \in S_{a_1 a_2 \dots a_{k-1} 1}$ as seen in Figure 6 (we use the abbreviation $\sigma = a_1 a_2 \dots a_{k-1}$ for simplicity). Note also that, in the other cases, i.e. a and b are in another sub-triangle of $S_{a_1 a_2 \dots a_{k-1}}$, similar procedures would be valid.

Let $p_\sigma, r_\sigma, q_\sigma$ be the intersection points of the sub-triangles $S_{\sigma 0}$ and $S_{\sigma 1}$, $S_{\sigma 0}$ and $S_{\sigma 2}$, $S_{\sigma 1}$ and $S_{\sigma 2}$ respectively. The shortest paths between a and b must pass through either the point p_σ or the line $r_\sigma q_\sigma$ (see Figure 6).

We now investigate these two different ways as follows:

Case 1: First consider the shortest path passing through the point p_σ . Any path between a and b can be expressed as the union of a path between a and p_σ and a path between p_σ and b . We first look at the shortest paths between a and p_σ (The paths between p_σ and b can be obtained using similar argument).

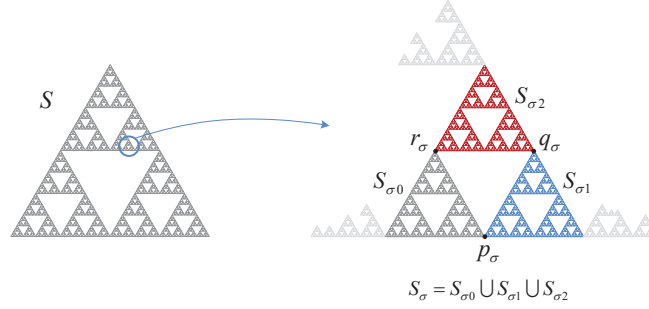


Figure 6. The sub-triangle S_σ where $\sigma = a_1 a_2 \dots a_{k-1}$ and the points $a \in S_{\sigma 0}$ and $b \in S_{\sigma 1}$.

• If $a \in S_{a_1 a_2 \dots a_{k-1} 00}$ or $a \in S_{a_1 a_2 \dots a_{k-1} 02}$ then we must compute the length of the line segment $p_{\sigma'} p_\sigma$ or the length of the line segment $q_{\sigma'} p_\sigma$ where $p_{\sigma'}, q_{\sigma'}$ are the intersection points of the sub-triangles $S_{\sigma'0}$ and $S_{\sigma'1}$, $S_{\sigma'1}$ and $S_{\sigma'2}$ respectively where $\sigma' = a_1 a_2 \dots a_{k-1} 0$ (see Figure 7). In the both cases, the length of the shortest paths between a and p_σ is

$$\mu = \frac{1}{2^{k+1}} + \varepsilon,$$

for some $\varepsilon \geq 0$.

For the case $a = r_{\sigma'}$, where $r_{\sigma'}$ is the intersection point of the sub-triangles $S_{\sigma'0}$ and $S_{\sigma'2}$, there exist obviously two shortest paths between a and p_σ (see Figure 7). These paths are the union of the line segments $r_{\sigma'} p_{\sigma'}$ and $p_{\sigma'} p_\sigma$ or the union of the line segments $r_{\sigma'} q_{\sigma'}$ and $q_{\sigma'} p_\sigma$. The length of these paths can be easily computed as $\mu = \frac{1}{2^k}$.

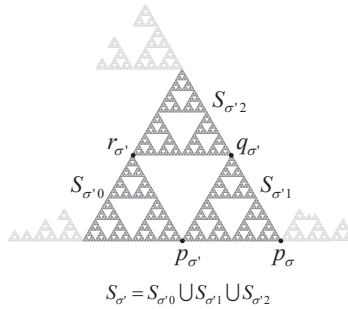


Figure 7. The sub-triangle $S_{\sigma'}$ where $\sigma' = a_1 a_2 \dots a_{k-1} 0$ and the points $a \in S_{\sigma'0}$ or $b \in S_{\sigma'2}$.

• Suppose that $a \in S_{a_1 a_2 \dots a_{k-1} 01}$.

If $a \in S_{a_1 a_2 \dots a_{k-1} 010}$ or $a \in S_{a_1 a_2 \dots a_{k-1} 012}$, then we must compute the length of the line segment $p_{\sigma''} p_\sigma$ or the length of the line segment $q_{\sigma''} p_\sigma$ where $p_{\sigma''}, q_{\sigma''}$

are the intersection points of the sub-triangles $S_{\sigma''0}$ and $S_{\sigma''1}$, $S_{\sigma''1}$ and $S_{\sigma''2}$ respectively where $\sigma'' = a_1 a_2 \dots a_{k-1} 01$ (see Figure 8).

In the both cases, we get

$$\mu = \frac{1}{2^{k+2}} + \varepsilon,$$

for some $\varepsilon \geq 0$.

For the case $a = r_{\sigma''}$, where $r_{\sigma''}$ is the intersection point of the sub-triangles $S_{\sigma''0}$ and $S_{\sigma''2}$, there are two paths giving the distance of the shortest paths between a and p_{σ} as were before. These paths are the union of the line segments $r_{\sigma''}p_{\sigma''}$ and $p_{\sigma''}p_{\sigma}$ or the union of the line segments $r_{\sigma''}q_{\sigma''}$ and $q_{\sigma''}p_{\sigma}$. The length of these two paths is $\mu = \frac{1}{2^{k+1}}$.

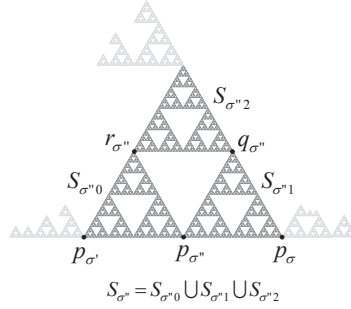


Figure 8. The sub-triangle $S_{\sigma''}$ where $\sigma'' = a_1 a_2 \dots a_{k-1} 01$.

Using similar procedure for smaller triangles, we can determine the shortest paths between a and b and the length of these paths. Similarly one can determine the shortest paths between p_{σ} and b . Then by splicing these shortest paths, between “ a and p_{σ} ” and “ p_{σ} and b ”, one can compute the length of the shortest paths between a and b passing through the point p_{σ} .

Case 2: Let us consider the shortest paths passing through the line segment $r_{\sigma}q_{\sigma}$. In a similar way, we can obtain the shortest paths (thus the corresponding length) between “ a and r_{σ} ” and between “ b and q_{σ} ”. As we add $\frac{1}{2^k}$ (that is, the length of the path $r_{\sigma}q_{\sigma}$) to these length, we obtain the length of the shortest path passing through $r_{\sigma}q_{\sigma}$.

Consequently, the length of the shortest paths between a and b is the minimum of the lengths obtained from Case 1 and Case 2. We can formulate this length (so the metric) as follows:

3.1. Definition. Let $a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots$ and $b_1 b_2 \dots b_{k-1} b_k b_{k+1} \dots$ be two representations respectively of the points $a \in S$ and $b \in S$ such that $a_i = b_i$ for $i = 1, 2, \dots, k-1$ and $a_k \neq b_k$. We define the distance $d(a, b)$ between a and b as

$$d(a, b) = \min \left\{ \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i} \right\}$$

where

$$\begin{aligned}
\alpha_i &= \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \\
\beta_i &= \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases}, \\
\gamma_i &= \begin{cases} 0, & a_i \neq a_k \text{ and } a_i \neq b_k \\ 1, & \text{otherwise} \end{cases}, \\
\delta_i &= \begin{cases} 0, & b_i \neq b_k \text{ and } b_i \neq a_k \\ 1, & \text{otherwise} \end{cases}.
\end{aligned}$$

3.2. Remark. Note that the first value $\sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}$ is the length of the shortest paths passing through the point p_σ and the second value $\frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i}$ is the length of the shortest paths passing through the line segment $r_\sigma q_\sigma$ where $\frac{1}{2^k}$ is the length of the line segment $r_\sigma q_\sigma$.

3.3. Proposition. *The distance function d defined in Definition 3.1 is strictly intrinsic metric on S .*

Proof. It is obvious from the fact that $d(a, b)$ is defined as the minimum of the lengths of the admissible paths connecting the points a and b in S . \square

3.4. Proposition. *The metric d defined in Definition 3.1 does not depend on the choice of the code representations of the points.*

Proof. Let a be a junction point whose code representations are of the form $a_1 a_2 a_2 \dots a_2 a_2 a_2 \dots$ and $a_2 a_1 a_1 \dots a_1 a_1 a_1 \dots$ such that $a_1 \neq a_2$ (in the general case, i.e. if the code representation of a is of the form $a_1 a_2 \dots a_{k-1} a_k a_{k+1} a_{k+1} a_{k+1} \dots$, the claim can be proven similarly).

Let x be an arbitrary point of S which has the code representation

$$x_1 x_2 \dots x_{k-1} x_k x_{k+1} x_{k+2} x_{k+3} \dots$$

Assume that $x_1 \neq a_1$. In this case, it must be $x_1 \neq a_2$ or $x_1 = a_2$.

Case 1: We first take $x_1 \neq a_2$. We now investigate the distance between the points

$$x_1 x_2 \dots x_k x_{k+1} x_{k+2} x_{k+3} \dots \text{ and } a_1 a_2 a_2 \dots a_2 a_2 a_2 \dots$$

Due to the definition of d , we have the following equations:

$$\begin{aligned}
\alpha_i &= \begin{cases} 0, & x_i = a_1 \\ 1, & x_i \neq a_1 \end{cases}, \\
\beta_i &= \begin{cases} 0, & a_2 = x_1 \\ 1, & a_2 \neq x_1 \end{cases}, \\
\gamma_i &= \begin{cases} 0, & x_i \neq x_1 \text{ and } x_i \neq a_1 \\ 1, & \text{otherwise} \end{cases},
\end{aligned}$$

$$\delta_i = \begin{cases} 0, & a_2 \neq a_1 \text{ and } a_2 \neq x_1 \\ 1, & \text{otherwise} \end{cases}.$$

We thus get $\beta_i = 1$ for all $i \geq 2$ owing to the fact that $x_1 \neq a_2$. Moreover, α_i can change according to the value of x_i and a_1 for each $i \geq 2$. It is also easily seen that $\delta_i = 0$ for every $i \geq 2$ since $a_2 \neq a_1$ and $a_2 \neq x_1$. It follows that

$$\sum_{i=2}^{\infty} \frac{\alpha_i + \beta_i}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{\alpha_i}{2^i}$$

and

$$\frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma_i + \delta_i}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma_i}{2^i}.$$

Now we compute the distance between the points

$$x_1 x_2 \dots x_k x_{k+1} x_{k+2} x_{k+3} \dots \text{ and } a_2 a_1 a_1 \dots a_1 a_1 a_1 \dots$$

Thanks to the definition of d , we have the following equations:

$$\begin{aligned} \alpha'_i &= \begin{cases} 0, & x_i = a_2 \\ 1, & x_i \neq a_2 \end{cases}, \\ \beta'_i &= \begin{cases} 0, & a_1 = x_1 \\ 1, & a_1 \neq x_1 \end{cases}, \\ \gamma'_i &= \begin{cases} 0, & x_i \neq x_1 \text{ and } x_i \neq a_2 \\ 1, & \text{otherwise} \end{cases}, \\ \delta'_i &= \begin{cases} 0, & a_2 \neq a_1 \text{ and } a_1 \neq x_1 \\ 1, & \text{otherwise} \end{cases}. \end{aligned}$$

Similarly, we have $\beta'_i = 1$ for all $i \geq 2$ owing to the fact that $x_1 \neq a_1$. Moreover, α'_i can change according to the value of x_i and a_2 for each $i \geq 2$. It is also obviously seen that $\delta'_i = 0$ for every $i \geq 2$ since $a_1 \neq a_2$ and $a_1 \neq x_1$. This shows that

$$\sum_{i=2}^{\infty} \frac{\alpha'_i + \beta'_i}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{\alpha'_i}{2^i}$$

and

$$\frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma'_i + \delta'_i}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma'_i}{2^i}.$$

Finally we show that $\alpha_i = \gamma'_i$ and $\alpha'_i = \gamma_i$ for all $i \geq 2$ respectively. We have already known that $a_1 \neq a_2$, $x_1 \neq a_1$ and $x_1 \neq a_2$.

Assume that $\gamma'_i = 0$ for a fixed i . In this case, we have $x_i \neq a_2$ and $x_i \neq x_1$. We thus have $x_i = a_1$. Namely, it is $\alpha_i = 0$. Let $\gamma'_i = 1$ for a fixed i . Hence it must be $x_i = a_2$ or $x_i = x_1$. This shows that $x_i \neq a_1$. That is we obtain $\alpha_i = 1$.

Suppose that $\gamma_i = 0$ for a fixed i . We thus have $x_i \neq x_1$ and $x_i \neq a_1$ and this shows that $x_i = a_2$. So we get $\alpha'_i = 0$. Let $\gamma'_i = 1$ for a fixed i . Therefore it must be $x_i = x_1$ or $x_i = a_1$. It follows that $x_i \neq a_1$ and thus we get $\alpha'_i = 1$.

This concludes the proof in Case 1.

Case 2: Let $x_1 = a_2$. The assertion can be proved similarly as above. □

4. Some Examples

In this section we give two examples in which we compute the distance between two kinds of pair of points in S .

4.1. Example. Let a and b be the points in S whose representations are $\overline{012} = 012012012 \dots$ and $\bar{1} = 111 \dots$ respectively (see Figure 9 for the place of the points).

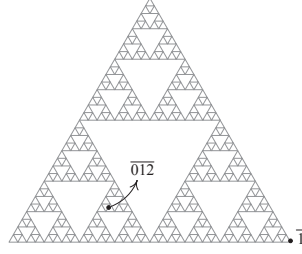


Figure 9. The points a and b coded by $\overline{012}$ and $\bar{1}$ respectively.

To compute $d(a, b)$ we need the natural number k defined in (3.1). Since the first term of the representations are different, we get $k = 1$. Easy calculations give us $\beta_i = 1, \delta_i = 1$,

$$\alpha_i = \begin{cases} 0 & ; \quad i \equiv 2 \pmod{3} \\ 1 & ; \quad \text{otherwise} \end{cases}$$

and

$$\gamma_i = \begin{cases} 0 & ; \quad i \equiv 0 \pmod{3} \\ 1 & ; \quad \text{otherwise} \end{cases}$$

for all $i \geq k + 1 = 2$. We then obtain

$$\sum_{i=2}^{\infty} \frac{\alpha_i + \beta_i}{2^i} = \sum_{m=1}^{\infty} \left(\frac{1}{2^{3m-1}} + \frac{2}{2^{3m}} + \frac{2}{2^{3m+1}} \right) = \frac{5}{7}$$

and

$$\frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma_i + \delta_i}{2^i} = \frac{1}{2} + \sum_{m=1}^{\infty} \left(\frac{2}{2^{3m-1}} + \frac{1}{2^{3m}} + \frac{2}{2^{3m+1}} \right) = \frac{1}{2} + \frac{6}{7}$$

which says that $d(a, b)$ is the minimum value $\frac{5}{7}$.

4.2. Example. Let a and b be the points in S whose representations are $000\bar{2} = 00022222 \dots$ and $0122\bar{0} = 012200000 \dots$ respectively (see Figure 10 for the place of the points).

Since the second term of the representations are different, we get $k = 2$. One can obtain $\alpha_i = 1$ for $i \geq k + 1 = 3$, $\beta_3 = \beta_4 = 1$ and $\beta_i = 0$ for $i \geq 5$, $\gamma_3 = 1$ and $\gamma_i = 0$ for $i \geq 4$, $\delta_3 = \delta_4 = 0$ and $\delta_i = 1$ for $i \geq 5$. We then obtain

$$\sum_{i=3}^{\infty} \frac{\alpha_i + \beta_i}{2^i} = \frac{2}{2^3} + \frac{2}{2^4} + \sum_{i=5}^{\infty} \frac{1}{2^i} = \frac{7}{16}$$

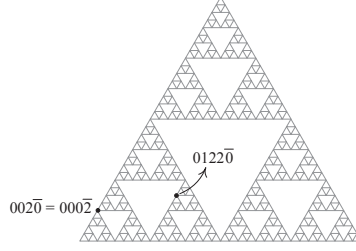


Figure 10. The points a and b coded by $000\bar{2}$ and $0122\bar{0}$ respectively.

and

$$\frac{1}{2^2} + \sum_{i=3}^{\infty} \frac{\gamma_i + \delta_i}{2^i} = \frac{1}{4} + \frac{1}{2^3} + \sum_{i=5}^{\infty} \frac{1}{2^i} = \frac{7}{16}$$

which says that $d(a, b)$ is the value $\frac{7}{16}$. Notice that two values are equal and it means that there exist at least two shortest paths between the points.

Indeed, since it is a junction point, the point $000\bar{2}$ has two code representations and one can take the representation of this point as $002\bar{0}$. In this case the computation yields $k = 2$, $\alpha_i = 1$ for $i \geq 3$, $\beta_3 = \beta_4 = 1$ and $\beta_i = 0$ for $i \geq 5$, $\gamma_3 = 0$ and $\gamma_i = 1$ for $i \geq 4$, $\delta_3 = \delta_4 = 0$ and $\delta_i = 1$ for $i \geq 5$. We then get by easy calculation $d(a, b) = \frac{7}{16}$ again as mentioned in Proposition 3.4.

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